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LETTER TO THE EDITOR

Relating different Poisson brackets of the KdV system to various symmetry conditions of the model

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Abstract. Three different Poisson bracket (PB) structures are known to exist for the KdV system. Using the non-ultralocal canonical property through the r, s-matrix formalism we are able to show that these PBs are originated from different symmetry conditions of the model, e.g. periodicity, antiperiodicity and aperiodicity.

Interest in the Hamiltonian structure of the KdV system has been revived recently [1–3]. The conventional Poisson bracket (PB) of this model was suggested long ago by Gardner, Zakharov and Faddeev which for functionals $F[u]$ and $G[u]$ may be given by

$$\{F, G\}_{\mathrm{as}} = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x \left(\frac{\delta F}{\delta u(x)} \left(\frac{\delta G}{\delta u(x)} \right)_x - \left(\frac{\delta F}{\delta u(x)} \right)_x \frac{\delta G}{\delta u(x)} \right). \tag{1}$$

Thereafter quite recently Faddeev and Takhtajan (FT) pointed out that the PB used earlier does not satisfy Jacobi identity and proposed an extension of it with a 'surface' term:

$$\{F, G\}_{\mathrm{FT}} = \{F, G\}_{\mathrm{as}} + \tfrac{1}{2}[F_+ G_- - F_- G_+] \tag{2}$$

with $F_\pm = \delta F/\delta u(x)|_{x\to\pm\infty}$. Subsequently another extension of the PB was put forward by Arkadiev *et al* (APP) to remove the degeneracy of the FT bracket:

$$\{F, G\}_{\mathrm{APP}} = \{F, G\}_{\mathrm{as}} - \tfrac{1}{2}[F_+ G_- - F_- G_+]. \tag{3}$$

We aim here to explore some possible underlying symmetry behind the existence of these *three* different PB structures in the *same* system. We analyse the given problem through an approach, which manifestly exploits the integrability of the system using the matrix Lax operator and the non-ultralocal PB structure (containing the derivative of δ-function) of the KdV system. This formalism, based on the classical r-matrix method, was first suggested by Tsyplyaev [4] and later developed considerably by Maillet [5] with the introduction of a novel s-matrix. Through a little extension of the above approach to infinite interval by introducing $s_\pm$ matrices we derive the PB relations between elements of the monodromy and scattering matrices in a compact matrix form.

Referring to [5] for detail discussion of the r, s-matrix method we present here only a few steps along with our result relevant to the KdV system. The Lax operator of the KdV system may be given in the AKNS form

$$U(z, \lambda) = \mathrm{i}(\lambda\sigma^3 - u(z)\sigma^- + \sigma^+)$$

with the non-ultralocal PB relation of the fundamental field: $\{u(x), u(y)\} = (\delta_x(x-y) - \delta_y(x-y))/2$, leading to the relation

$$\{U(z,\lambda) \underset{,}{\otimes} U(z',\mu)\} = [\delta_{z'}(z-z') - \delta_z(z-z')]\sigma^- \otimes \sigma^-/2.$$

For the related monodromy matrix defined over some finite interval as $T_y^x(\lambda) = \mathcal{P}\exp\int_y^x U(x',\lambda)\,dx'$, one gets through an intermediate step

$$\{T_y^x(\lambda) \underset{,}{\otimes} T_{y'}^{x'}(\mu)\} = \int_y^x dz \int_{y'}^{x'} dz'\, T_z^x(\lambda) \otimes T_{z'}^{x'}(\mu)\{U(z,\lambda) \underset{,}{\otimes} U(z',\mu)\} T_y^z(\lambda) \otimes T_{y'}^{z'}(\mu) \tag{4}$$

the relevant PB relation

$$\begin{aligned}\{T_y^x(\lambda) \underset{,}{\otimes} T_{y'}^{x'}(\mu)\} &= T_{x_0}^x(\lambda) \otimes T_{x_0}^{x'}(\mu)[r_L(\lambda,\mu) + \operatorname{sgn}(x-x')s_L] T_y^{x_0}(\lambda) \otimes T_{y'}^{x_0}(\mu) \\ &\quad - T_{y_0}^x(\lambda) \otimes T_{y_0}^{x'}(\mu)[r_L(\lambda,\mu) - \operatorname{sgn}(y-y')s_L] T_y^{y_0}(\lambda) \otimes T_{y'}^{y_0}(\mu)\end{aligned} \tag{5}$$

where $x_0 = \min(x, x')$, $y_0 = \max(y, y')$ and r_L, s_L are given by the form

$$\begin{aligned} r_L(\lambda,\mu) &= -[(\sigma^3 \otimes \sigma^- - \sigma^- \otimes \sigma^3) - (\lambda-\mu)\sigma^- \otimes \sigma^- + P/(\lambda-\mu)]/2(\lambda+\mu) \\ s_L &= \tfrac{1}{2}\sigma^- \otimes \sigma^- \end{aligned} \tag{6}$$

with $P = \frac{1}{2}(1 \otimes 1 + \Sigma_i \sigma^i \otimes \sigma^i)$. For investigating the effect of non-ultralocality we insert the particular cases (i) $x = 3L$, $y = x' = L$, $y' = -L$ and (ii) $x = y' = -L$, $y = -3L$, $x' = L$ in the general formula (5) to obtain

$$\{T_L^{3L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\} = (T_L^{3L}(\lambda) \otimes 1)s_L(1 \otimes T_{-L}^{L}(\mu)) \tag{7a}$$

$$\{T_{-3L}^{-L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\} = -(1 \otimes T_{-L}^{L}(\mu))s_L(T_{-3L}^{-L}(\lambda) \otimes 1). \tag{7b}$$

Note that the non-vanishing PB for adjacent intervals as found here reflects explicitly the non-ultralocal property of the system. On the other hand using the property $T_{-3L}^{3L}(\lambda) = T_L^{3L}(\lambda)T_{-L}^{L}(\lambda)T_{-3L}^{-L}(\lambda)$ we get

$$\begin{aligned}\{T_{-3L}^{3L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\} &= \{T_L^{3L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\}(T_{-3L}^{L}(\lambda) \otimes 1) + (T_L^{3L}(\lambda) \otimes 1)\{T_{-L}^{L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\} \\ &\quad \times (T_{-3L}^{-L}(\lambda) \otimes 1) + (T_{-L}^{3L}(\lambda) \otimes 1)\{T_{-3L}^{-L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\}\end{aligned}$$

which, with the use of (5) and (7*a*, *b*), simplifies to

$$\{T_{-L}^{L}(\lambda) \underset{,}{\otimes} T_{-L}^{L}(\mu)\} = r_L(\lambda,\mu)T_{-L}^{L}(\mu) - T_{-L}^{L}(\lambda) \otimes T_{-L}^{L}(\mu)r_L(\lambda,\mu). \tag{7c}$$

In order now to link the various Poisson bracket structures (1), (2) and (3) to the underlying symmetry of the model, we consider boundary conditions with different types of symmetry, e.g., periodic, antiperiodic and aperiodic ones. Periodic models with the property $u(x \pm 2L) = u(x)$ induce $T_{-L\pm 2L}^{L\pm 2L}(\lambda) = T_{-L}^{L}(\lambda)$. Consequently, for periodic models any of (7*a*), (7*b*), (7*c*) might be taken as the definition for PB given in the symmetric form as $\{\,,\,\}_{\rm Per} = (7a) + (7b) + (7c)$ leading to

$$\{T_L \underset{,}{\otimes} \tilde{T}_L\}_{\rm Per} = r_L T_L \otimes \tilde{T}_L - T_L \otimes \tilde{T}_L r_L + [(T_L \otimes 1)s_L(1 \otimes \tilde{T}_L) - (1 \otimes \tilde{T}_L)s_L(T_L \otimes 1)] \tag{8a}$$

with $T_L \equiv T_{-L}^{L}(\lambda)$ and $\tilde{T}_L \equiv T_{-L}^{L}(\mu)$.

In the antiperiodic model on the other hand, we have $u(x \pm 2L) = -u(x)$, which from expression $T_{-L\pm 2L}^{L\pm 2L} = \mathcal{P}\exp\int_{-L\pm 2L}^{L\pm 2L} U(u(x),\lambda)\,dx$ clearly yields $T_{-L\pm 2L}^{L\pm 2L}[u,\lambda] = T_{-L}^{L}[-u,\lambda]$ after a shift in the integration variable. Therefore, for deriving the PB between the monodromy matrices $T^L_{-L}[u,\lambda]$ from (7*a*) and (7*b*) we have to flip the sign of $u(x)$ in $T^{L\pm 2L}_{-L\pm 2L}[u,\lambda]$. However, since these PBs are proportional to u as seen from (4), it induces a change in the sign of PB (7*a*, *b*) leading to the definitions of $\{\ ,\ \}_{\text{Antiper}} = (7c) - ((\tilde{7}a) + (\tilde{7}b))$ in the antiperiodic case, where $(\tilde{7}a)$, $(\tilde{7}b)$ indicates replacing $T^{L\pm 2L}_{-L\pm 2L}[u,\lambda]$ by $T^L_{-L}[-u,\lambda]$ on the RHS of the respective equations. This ultimately results in

$$\{T_L \underset{,}{\otimes} \tilde{T}_L\}_{\text{Antiper}} = r_L T_L \otimes \tilde{T}_L - T_L \otimes \tilde{T}_L r_L - [(T_L \otimes 1)s_L(1 \otimes \tilde{T}_L) - (1 \otimes \tilde{T}_L)s_L(T_L \otimes 1)]. \tag{8b}$$

Finally, in the aperiodic case, since T^L_{-L} and $T^{L\pm 2L}_{-L\pm 2L}$ at adjacent intervals are no longer related, one may take (7*c*) alone as the definition of the corresponding PB yielding

$$\{T_L \underset{,}{\otimes} \tilde{T}_L\}_{\text{Aper}} = r_L T_L \otimes \tilde{T}_L - T_L \otimes \tilde{T}_L r_L. \tag{8c}$$

We consider now the infinite interval limit of these brackets, since PBs (1), (2), (3) for the KdV system are defined essentially in this limit. Defining the scattering matrix as [6]

$$T(\lambda) = \lim_{\substack{x\to+\infty \\ y\to-\infty}} E_+^{-1}(x,\lambda) T^x_y(\lambda) E_-(y,\lambda) = \begin{pmatrix} \bar{a}(\lambda) & b(\lambda) \\ \bar{b}(\lambda) & a(\lambda) \end{pmatrix} \tag{9}$$

with the choice of asymptotic solutions

$$E_\pm = \begin{pmatrix} e^{i\lambda x} & e^{-i\lambda x} \\ 0 & -2\lambda\, e^{-i\lambda x} \end{pmatrix}$$

one may shift now to the infinite interval limit of (8*a*, *b*, *c*). Thus we arrive at the relations

$$\{T \underset{,}{\otimes} \tilde{T}\}_{\substack{\text{Per} \\ (\text{Antiper})}} = [r_+ T \otimes \tilde{T} - T r_-] \pm [(T \otimes 1)s_+(1 \otimes \tilde{T}) - (1 \otimes \tilde{T})s_-(T \otimes 1)] \tag{10a, b}$$

$$\{T \underset{,}{\otimes} \tilde{T}\}_{\text{Aper}} = [r_+ T \otimes \tilde{T} - T \otimes \tilde{T} r_-] \tag{10c}$$

where $T \equiv T(\lambda)$, $\tilde{T} \equiv T(\lambda)$ and

$$r_\pm(\lambda,\mu) = -\alpha\sigma_3 \otimes \sigma_3 \pm f_1 \sigma_3 \otimes \sigma_1 \mp f_2 \sigma_1 \otimes \sigma_3 \mp i f_3(\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1) 2 \mp i f_4 (\sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1)/2$$

$$s_\pm(\lambda,\mu) = \frac{1}{8\lambda\mu} I \otimes I \mp f_1 \sigma_3 \otimes \sigma_1 \pm f_2 \sigma_1 \otimes \sigma_3$$

with

$$\alpha = (\lambda^2 + \mu^2)/8\lambda\mu(\lambda^2 - \mu^2) \qquad f_1 = i\pi\delta(\mu)/8\lambda \qquad f_2 = i\pi\delta(\lambda)/8\mu$$

$$f_3 = i\pi(\lambda - \mu)\delta(\lambda + \mu)/8\lambda\mu$$

and

$$f_4 = i\pi(\lambda + \mu)\delta(\lambda - \mu)/8\lambda\mu.$$

The matrix equations (10*a*), (10*b*) and (10*c*) may now be expressed for the scattering matrix elements using (9). However for the KdV system, as is well known, the scattering elements may have a pole-type singularity at $\lambda \to 0$ of the form [7] $a(\lambda) \sim im/2\lambda$ and $b(\lambda) \sim -im/2\lambda$, where m is a constant of motion [8]. We shall consider here only the case when $m \neq 0$. Consideration of the said singularity structure in the equations (10*a*)-(10*c*) leads finally to the PB relations

$$\{\ln a(\lambda), \ln b(\mu)\} = f(\lambda,\mu,\varepsilon) - i\pi(\delta(\lambda-\mu) - \delta(\lambda+\mu))/4\lambda + (1+\varepsilon) i\pi\delta(\lambda)/4\mu$$

$$\{\ln a(\lambda), \ln a(\mu)\} = 0$$

with $f(\lambda, \mu, \varepsilon) = (1/4\lambda\mu)[(\lambda^2+\mu^2)/(\lambda^2-\mu^2)-\varepsilon]$, where $\varepsilon = 0$ corresponds to the aperiodic, $\varepsilon = +1$ to the periodic and $\varepsilon = -1$ to the antiperiodic cases.

It is surprising to note that these PBs for the scattering matrix elements coincide exactly with those obtained from the definitions (1), (2) and (3) [1, 3] and moreover, the periodic case ($\varepsilon = 1$) corresponds to the Faddeev-Takhtajan bracket, antiperiodic case ($\varepsilon = -1$) to the Arkadiev *et al* bracket, while the aperiodic case ($\varepsilon = 0$) yields the Gardner-Zakharov-Faddeev bracket. Therefore the underlying symmetry behind the existence of three different PBs of the KdV system, which is not evident at the infinite interval limit, becomes explicit at the finite interval and relates different PB structures to it.

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